Alternate Form of the Cubic Formula

We want to find the three solutions to

$$x^3 + px^2 + qx + r = 0. \qquad (1)$$

In what follows, let

$$c = \tfrac{1}{9}(3q - p^2), \qquad d = \tfrac{1}{54}(27r - 9pq + 2p^3). \qquad (2)$$

The solutions to eqn. (1) follow three cases.

Case 1; $c = 0$ and $d = 0$:

In this case, there is one real root (of multiplicity three). Namely,

$$x = -\tfrac{1}{3}p \quad \text{(three times).} \qquad (3)$$

Case 2; $c = 0$ and $d \neq 0$:

In this case, there will be one real root and one pair of complex conjugate roots. Let

$$A^3 = -2d. \qquad (4)$$

Next, find three values of A which are the three cube roots of A^3 in eqn. (4). The three solutions to eqn. (1) are then obtained by substituting, in turn, these three values of A into

$$x = -\tfrac{1}{3}p + A. \qquad (5)$$

Case 3; $c \neq 0$:

Let

$$A^3 = -d - \sqrt{d^2 + c^3}\,, \qquad (6)$$

in which the square root is the principal square root. Next, find three values of A which are the three cube roots of A^3 in eqn. (6). The three solutions to eqn. (1) are then obtained by substituting, in turn, these three values of A into

$$x = -\tfrac{1}{3}p + A - \tfrac{c}{A}\ . \qquad (7)$$

In this case, if

a) $d^2 + c^3 > 0$ (so that A^3 is real), then there will be one real root and one pair of complex conjugate roots.

b) $d^2 + c^3 = 0$ (so that A^3 is real), then there will be one real root (of multiplicity one) and another real root (of multiplicity two).

c) $d^2 + c^3 < 0$ (so that A^3 is complex), then there will be three distinct real roots.