ALTERNATE FORM OF THE CUBIC FORMULA – DERIVATION

1. The Hyperbolic Sine Function

Before proceeding, a few facts about the hyperbolic sine function are needed. Consequently, the hyperbolic sine is defined by

$$\sinh t = \tfrac{1}{2}(e^t - e^{-t}) \,. \tag{1}$$

Now, from eqn. (1), the identity

$$\sinh 3t = 4\sinh^3 t + 3\sinh t \tag{2}$$

may be verified. The inverse function of eqn. (1) is also required. Thus,

$$u = \sinh t \,, \qquad t = \sinh^{-1} u \,. \tag{3}$$

To find the inverse, multiply eqn. (1) by e^t to obtain

$$0 = (e^t)^2 - 2ue^t - 1 \,, \tag{4}$$

where the first of eqns. (3) was used. The Quadratic Formula then gives

$$e^t = u \pm \sqrt{u^2+1} \tag{5}$$

or

$$t = \ln\left(u \pm \sqrt{u^2+1}\right) . \tag{6}$$

Finally, via the second of eqns. (3), one has

$$\sinh^{-1} u = \ln\left(u \pm \sqrt{u^2+1}\right) . \tag{7}$$

2. An Alternate Form of the Cubic Formula

We want to find the three solutions to

$$P(x) = x^3 + px^2 + qx + r = 0 \,, \tag{8}$$

where p, q and r are real. Pursuantly, substitution of

$$x = -\tfrac{1}{3}p + y \tag{9}$$

into eqn. (8) yields

$$P(y) = y^3 + 3cy + 2d = 0 \,, \tag{10}$$

where

$$c = \tfrac{1}{9}(3q - p^2) \,, \qquad d = \tfrac{1}{54}(27r - 9pq + 2p^3) \,. \tag{11}$$

Next, substitute

$$y = 2\sqrt{c}\sinh z \tag{12}$$

into eqn. (10) to obtain

$$\left(\sqrt{c}\right)^3 (4\sinh^3 z + 3\sinh z) + d = 0 \,, \tag{13}$$

or via eqn. (2),

$$\sinh 3z = -\frac{d}{\left(\sqrt{c}\right)^3} \,. \tag{14}$$

Inverting eqn. (14) then gives

$$z = \tfrac{1}{3}\sinh^{-1}\left[-\frac{d}{\left(\sqrt{c}\right)^3}\right] . \tag{15}$$

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Now, by using eqn. (7), eqn. (15) becomes

$$z = \frac{1}{3}\ln\left[\left(\frac{A}{\sqrt{c}}\right)^3\right] = \ln\left(\frac{A}{\sqrt{c}}\right), \tag{16}$$

where

$$A^3 = -d - \sqrt{d^2 + c^3}\,. \tag{17}$$

Note that, in eqn. (17), the negative branch of eqn. (7) has been chosen. Notwithstanding, proceed by putting eqn. (16) into eqn. (12), and then by using eqn. (1). Whence,

$$y = A - \frac{c}{A}\,. \tag{18}$$

Finally, substitution of eqn. (18) into eqn. (9) solves the problem at hand. Namely,

$$x = -\frac{1}{3}p + A - \frac{c}{A}\,. \tag{19}$$

3. Method of Solution

First, use eqns. (11) to obtain c and d. Second, calculate A^3 by way of eqn. (17). Next, find three values for A by calculating the three cube roots of A^3. Finally, substitute, in turn, these three values of A into eqn. (19) to obtain three values of x. The three values of x so-obtained are the three roots of eqn. (8).

4. Degenerate Cases

If $c = 0$, eqn. (10) is

$$y^3 = -2d\,, \tag{20}$$

which yields three values of y, *i.e.*, the three cube roots of y^3. The three roots of eqn. (8) are then obtained by substituting, in turn, these three values of y into eqn. (9).

Finally, if $c = 0$ and $d = 0$, eqn. (10) is $y^3 = 0$, or $y = 0$ (three times). In this case, then, via eqn. (9), $x = -(1/3)p$ three times, *i.e.*, here eqn. (8) possesses a triple root.